THIRD ORDER SEMILINEAR DISPERSIVE EQUATIONS RELATED TO DEEP WATER WAVES

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ABSTRACT. We present local existence theorem of the initial value problem for third order semilinear dispersive partial differential equations in two space dimensions. This type of equations arises in the study of gravity wave of deep water, and cannot be solved by the classical energy method. To solve the initial value problem, we make full use of pseudodifferential operators with nonsmooth coefficients.

1. INTRODUCTION

In this paper we study the initial value problem for third order semilinear dispersive equations of the form

$$\partial_t u + p(\partial)u = \sum_{j=0}^{3} a_j f_j(u) \quad \text{in} \quad \mathbb{R}^{1+2}, \tag{1}$$

$$u(0,x) = u_0(x) \qquad\qquad \text{in} \quad \mathbb{R}^2, \tag{2}$$

where $u(t,x)$ is a complex-valued unknown function of $(t,x) = (t,x_1,x_2) \in \mathbb{R}^{1+2}$, $u_0(x)$ is an initial data, $\partial_t = \partial/\partial t$, $\partial_j = \partial/\partial x_j$, $\partial = (\partial_1, \partial_2)$, $p(i\xi)$ is a pure-imaginary polynomial of degree three of $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$, $i = \sqrt{-1}$, $a_j \in \mathbb{C}$,

$$f_0(u) = uR_1\partial_1|u|^2, f_1(u) = |u|^2\partial_1 u, f_2(u) = u^2\partial_1\bar{u}, f_3 u = |u|^2 u,$$

$R_1 = \partial_1(-\Delta)^{-1/2}$ and $\Delta = \partial_1^2 + \partial_2^2$.

This type of partial differential equations arises in the study of gravity wave of deep water. In fact,

$$\left(\partial_t - \frac{1}{16}(\partial_1^3 - 6\partial_1\partial_2^2) + \frac{i}{8}(\partial_1^2 - 2\partial_2^2) + \frac{1}{2}\partial_1\right)u$$
$$= -\frac{i}{2}f_0(u) - \frac{3}{2}f_1(u) + \frac{1}{4}f_2(u) - \frac{i}{2}f_3(u)$$

was derived by Dysthe in [7], and

$$\left(\partial_t - (b_1\partial_1^3 + b_2\partial_1\partial_2^2) + i(b_3\partial_1^2 + b_4\partial_2^2) + b_5\partial_1\right)u$$
$$= -\frac{i}{2}f_0(u) + \mu_1 f_1(u) + \mu_2 f_2(u) + i\mu_3 f_3(u),$$
$$b_1, b_2, b_3, b_4, b_5, \mu_1, \mu_2, \mu_3 \in \mathbb{R},$$

was formulated by Hogan in [8]. Note that a_0 and a_3 are pure-imaginary, and a_1 and a_2 are real in the above physical models. If $a_0 \neq 0$ or $\operatorname{Im} a_1 \neq 0$, then the loss of derivatives occurs in (1), and the classical energy method does not work. More precisely, $(\operatorname{Re} a_0)|u|^2 R_1\partial_1 u$ in $a_0 f_0(u)$, $a_0 u^2 R_1 \partial_1 \bar{u}$ in $a_0 f_0(u)$ and $(\operatorname{Im} a_1)|u|^2\partial_1 u$ in $a_1 f_1(u)$ cannot be controled by the classical energy method. $f_2(u)$ has no problem since $f_2(u)\bar{u} = u^2\partial_1(\bar{u}^2)/2$, which is controled by the integration by parts.

For this reason, there are few results on the existence of solutions to (1)-(2). In [4], using the abstract Cauchy-Kowalewski theorem, de Bouard proved time-local existence of a unique solution to the generalized equations of (1) for a real-analytic initial data. Recently, the time decay of the fundamental solution $e^{-tp(\partial)}$ was discussed by Ben-Artzi, Koch and Saut in [1].

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Let $p_0(\partial)$ be the principal part of $p(\partial)$, that is, $p_0(\xi)$ is a homogeneous polynomial of degree three, and $p(\xi)-p_0(\xi)$ is a polynomial of degree two. It is said that $e^{-tp(\partial)}$ has local smoothing effect if $e^{-tp(\partial)}$ gains one derivative in x locally in $\mathbb{R}^{1+2}$. More generelly, in case that $p(\partial)$ is an operator of order $m>1$ on $\mathbb{R}^n$, it is said that $e^{-tp(\partial)}$ has local smoothing effect if $e^{-tp(\partial)}$ gains $(-\Delta)^{(m-1)/4}$ locally in $\mathbb{R}^{1+n}$. In [9], Hoshiro proved that $e^{-tp(\partial)}$ has local smoothing effect if and only if

$$p_0'(\xi)=\nabla_\xi p_0(\xi)\neq 0 \quad \text{for} \quad \xi\neq 0. \tag{3}$$

(3) is equivalent to so-called nontrapping condition of classical orbits

$$\lim_{t\to\pm\infty}|x+tp_0'(\xi)|=+\infty \quad \text{for} \quad (x,\xi)\in\mathbb{R}\times\mathbb{R}\setminus\{0\}.$$

For the detail of the smoothing effect of dispersive equations, see [3], [6], [9], [13] and references therein. Under the nontrapping condition, one can expect the standard existence theorem for (1)-(2).

The aim of this paper is to present time-local existence theorem of (1)-(2) in appropriate Sobolev spaces. Here we introduce notation regarding function spaces. Let $L^2(\mathbb{R}^2)$ be the set of all square-integrable functions on $\mathbb{R}^2$. For $f,g\in L^2(\mathbb{R}^2)$, set

$$(f,g)=\int_{\mathbb{R}^2}f(x)\overline{g(x)}dx, \quad \|f\|=\sqrt{(f,g)}.$$

Let $\langle D\rangle=(1-\Delta)^{1/2}$. For $s\in\mathbb{R}$, set $H^s(\mathbb{R}^2)=\langle D\rangle^{-s}L^2(\mathbb{R})$ and $\|f\|_s=\|\langle D\rangle^s f\|$. Let I be an interval in $\mathbb{R}$. $C(I;H^s(\mathbb{R}^2))$ is the set of all $H^s(\mathbb{R}^2)$-valued continuous functions on I. $L^1(I;H^s(\mathbb{R}^2))$ is the set of all $H^s(\mathbb{R}^2)$-valued integrable functions on I. $L^\infty(I;H^s(\mathbb{R}^2))$ is the set of all $H^s(\mathbb{R}^2)$-valued essentially bounded functions on I. In this paper $(\cdot,\cdot)$ and $\|\cdot\|$ sometimes mean the inner product and the norm of $\mathbb{C}^l$-valued functions respectively, that is,

$$(U,V)=\sum_{\nu=1}^{l}(u_\nu,v_\nu), \quad \|U\|^2=(U,U)$$

for $U=(u_1,\dots,u_l), V=(v_1,\dots,v_l)\in(L^2(\mathbb{R}^2))^l$. Any confusion will not occur. Here we state our results.

Theorem 1. *Suppose* (3) *and* $s>3$. *Then, for any* $u_0\in H^s(\mathbb{R}^2)$, *there exists* $T=T(\|u_0\|_s)>0$ *such that* (1)-(2) *possesses a unique solution* $u\in C([-T,T];H^s(\mathbb{R}^2))$.

Our method of proof of Theorem 1 is an energy method via pseudodifferential calculus. This method was developed in [2] for semilinear Schrödinger equations of the form

$$\partial_t u-i\Delta u=f(u,\partial u) \quad \text{in} \quad \mathbb{R}^{1+n}. \tag{4}$$

The basic idea comes from the theory of well-posedness of the initial value problem for linear dispersive equations. See, e.g., [3], [5], [14], [15] and references therein. In particular, Tarama obtained the necessary and sufficient condition for the L^2-well-posedness of the initial value problem for one dimensional third order equations. See [14] and [15] for the detail. To apply linear theory to nonlinear equations including ∂u, $\partial\bar{u}$ and so on, it is very convenient to consider the system of the original equation and its complex conjugate. In other words, we consider the system for ${}^t[u,\bar{u}]$, where u is an unknown function of the original nonlinear equation. To control the bad first order terms by the local smoothing effect, we make use of a pseudodifferential operator of order zero discovered by Doi in [5]. Using the unknown function u, we construct a nonsmooth symbol of the pseudodifferential operator. It is very important that the symbol is the type $(\rho,\delta)=(1,0)$, and the corresponding operator is easy to handle. (1)-(2) is easier than the initial value problem for (4) since the local smoothing effect of $e^{-tp(\partial)}$ is stronger than that of $e^{it\Delta}$, and the principal part of (1) and that of its complex conjugate are exactly same. In case of the semilinear Schrödinger equation (4), the principal parts of equations of u and $\bar{u}$ are $-i\Delta$ and $i\Delta$ respectively, and the diagonalization technique was used so that the smoothing estimates via pseudodifferential calculus worked. See [2] for the detail. If the principal part of (4) is not elliptic, then the diagonalization

does not work. In this case, an operator of the exotic class was used, and higher smoothness of the initial data was required. See [10] for the detail. Our symbolic calculus is based on the symbol smoothing method for pseudodifferential operators with nonsmooth coefficients. This method was discovered by Nagase in [12]. For the basic pseudodifferential calculus, see e.g., [11], [16], [17] and references therein.

The organization of this paper is as follows. In Section 2 we construct a sequence of approximate solutions by parabolic regularization. In Section 3 we summarize pseudodifferential calculus and related nonlinear estimates needed later. In Section 4 we study a linear system related to (1). In Section 5 we prove Theorem 1.

2. A SEQUENCE OF APPROXIMATE SOLUTIONS

In this section we construct a sequence of approximate solutions by parabolic regularization. Let ε be a positive parameter. Consider a parametrized initial value problem of the form

$$\partial_t u + p(\partial)u - \varepsilon\Delta u = \sum_{j=0}^{3} a_j f(u) \quad \text{in} \quad (0,+\infty)\times\mathbb{R}^2, \tag{5}$$

$$u(0,x) = u_0(x) \qquad \text{in} \quad \mathbb{R}^2. \tag{6}$$

Since $e^{t(-p(\partial)+\varepsilon\Delta)}$ can control first order terms locally in time, (5)-(6) has a time-local unique solution.

Lemma 2. *Suppose $s > 2$. Then, for any $u_0 \in H^s(\mathbb{R}^2)$, there exists $T_\varepsilon = T(\varepsilon, \|u_0\|_s) > 0$ such that* (5)-(6) *possesses a unique solution $u \in C([0,T_\varepsilon]; H^s(\mathbb{R}^2))$. Moreover, the map $u_0 \mapsto u$ is continuous.*

To prove Lemma 2, we need the following estimates.

Lemma 3. (i) *Let $s \in \mathbb{R}$. Then, for any $u \in H^s(\mathbb{R}^2)$ and $t > 0$,*

$$\|e^{t(-p(\partial)+\varepsilon\Delta)}u\|_s \leqslant C\left(1+\frac{1}{\sqrt{t\varepsilon}}\right)\|u\|_{s-1}.$$

(ii) *Let $s > 2$. Then, for any $j = 0,1,2,3$ and $u, v \in H^s(\mathbb{R}^2)$,*

$$\|f_j(u)\|_{s-1} \leqslant C\|u\|_s^3,$$
$$\|f_j(u)-f_j(v)\|_{s-1} \leqslant C(\|u\|_s^2+\|v\|_s^2)\|u-v\|_s.$$

Here we introduce notation. For a multi-index $\alpha = (\alpha_1, \alpha_2)$, set $|\alpha| = \alpha_1 + \alpha_2$, $\partial^\alpha = \partial_1^{\alpha_1}\partial_2^{\alpha_2}$ and $\xi^\alpha = \xi_1^{\alpha_1}\xi_2^{\alpha_2}$. Let $\sigma \geqslant 0$, and let $[\sigma]$ be the largest integer less than or equal to σ. $\mathscr{B}^\sigma(\mathbb{R}^2)$ is the set of all $C^{[\sigma]}$-functions $f(x)$ on $\mathbb{R}^2$ satisfying $\|f\|_{\mathscr{B}^\sigma} < +\infty$, where

$$\|f\|_{\mathscr{B}^\sigma} = \begin{cases} \displaystyle\sup_{x\in\mathbb{R}^2}\sum_{|\alpha|\leqslant\sigma}|\partial^\alpha f(x)| & (\sigma = 0,1,2,3,\dots) \\ \displaystyle\|f\|_{\mathscr{B}^{[\sigma]}} + \sup_{\substack{x,y\in\mathbb{R}^2\\ x\neq y}}\sum_{|\alpha|=[\sigma]}\frac{|\partial^\alpha f(x)-\partial^\alpha f(y)|}{|x-y|^{s-[s]}} & (\text{otherwise}) \end{cases}$$

The Fourier transform of $u(x)$ is defined by

$$\hat{u}(\xi) = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{-ix\cdot\xi}u(x)dx, \quad x\cdot\xi = x_1\xi_1 + x_2\xi_2.$$

Proof of Lemma 3. (i) Set $|\xi| = \sqrt{\xi_1^2+\xi_2^2}$ and $\langle\xi\rangle = \sqrt{1+|\xi|^2}$ for short. In view of the Plancherel-Perseval formula, we deduce

$$\begin{aligned}\|e^{t(-p(\partial)+\varepsilon\Delta)}u\|_s &= \|\langle\xi\rangle^s e^{t(-p(i\xi)-\varepsilon|\xi|^2)}\hat{u}\| \\ &\leqslant \sup_{\xi\in\mathbb{R}^2}\langle\xi\rangle e^{-t\varepsilon|\xi|^2}\|u\|_{s-1}\end{aligned}$$

$$\leqslant \left(1+\sup_{\xi\in\mathbb{R}^2}|\xi|e^{-t\varepsilon|\xi|^2}\right)\|u\|_{s-1}$$
$$\leqslant C\left(1+\frac{1}{\sqrt{t\varepsilon}}\right)\|u\|_{s-1}.$$

(ii) We show only the estimate of $f_0(u)$. Recall the Sobolev embedding $H^s(\mathbb{R}^2)\subset\mathscr{B}^\sigma(\mathbb{R}^2)$ for $s>1+\sigma$. It is easy to see that for $s>2$ and $u,v\in H^{s-1}(\mathbb{R})$,

$$\|uv\|_{s-1}\leqslant 2^{s-1}(\|u\|_{s-1}\|v\|_{\mathscr{B}^0}+\|u\|_{\mathscr{B}^0}\|v\|_{s-1})\leqslant C_s\|u\|_{s-1}\|v\|_{s-1}.$$

Using this estimate and the L^2-boundedness of R_1, we deduce

$$\begin{aligned}\|f_0(u)\|_{s-1}&\leqslant C_s\|u\|_{s-1}\|R_1\partial_1|u|^2\|_{s-1}\\&\leqslant C_s\|u\|_{s-1}\|\partial_1|u|^2\|_{s-1}\\&\leqslant 2C_s\|u\|_{s-1}\|\bar{u}\partial_1 u\|_{s-1}\\&\leqslant 2C_s^2\|u\|_{s-1}^2\|\partial_1 u\|_{s-1}\\&\leqslant 2C_s^2\|u\|_{s-1}^2\|u\|_s.\end{aligned}$$

The other estimates can be obtained by similar computation. We omit the detail. □

Applying the contraction mapping theorem and Lemma 3 to an integral equation

$$u(t)=e^{t(-p(\partial)+\varepsilon\Delta)}u_0+\int_0^t e^{(t-\tau)(-p(\partial)+\varepsilon\Delta)}\sum_{j=0}^3 a_j f_j(u(\tau))d\tau, \tag{7}$$

we can show Lemma 2. We omit the detail.

3. Pseudodifferential operators with nonsmooth coefficients

Following [2], we give the pseudodifferential calculus used later. We introduce symbols with non-smooth coefficients.

Definition 1. Let $m\in\mathbb{R}$ and $\sigma\geqslant 0$. $\mathscr{B}^\sigma\Psi^m(\mathbb{R}^n)$ is the set of all functions $p(x,\xi)$ on $\mathbb{R}^n\times\mathbb{R}^n$ satisfying

$$\|p\|_{\mathscr{B}^\sigma\Psi^m,l}=\sup_{\xi\in\mathbb{R}^n}\sum_{|\alpha|\leqslant l}\langle\xi\rangle^{m-|\alpha|}\|\partial_\xi^\alpha p(\cdot,\xi)\|_{\mathscr{B}^\sigma}<+\infty$$

for $l=0,1,2,3,\dots$.

Set $D=-i\partial$. For $p(x,\xi)\in\mathscr{B}^\sigma\Psi^m(\mathbb{R}^n)$, a pseudodifferential operator $p(x,D)$ is defined by

$$p(x,D)u(x)=\frac{1}{(2\pi)^n}\iint_{\mathbb{R}^n\times\mathbb{R}^n}e^{i(x-y)\cdot\xi}p(x,\xi)u(y)dyd\xi,$$

where $x\cdot\xi=x_1\xi_1+\cdots+x_n\xi_n$ for $x=(x_1,\cdots,x_n)$ and $\xi=(\xi_1,\dots,\xi_n)$. Conversely, if an operator P is given, then, its symbol is given by $\sigma(P)(x,\xi)=e^{-ix\cdot\xi}Pe^{ix\cdot\xi}$. The properties of the above pseudodifferential operators are the following.

Lemma 4. (i) *Suppose $\sigma>0$ and $p(x,\xi)\in\mathscr{B}^\sigma\Psi^0(\mathbb{R}^n)$. Then, there exist $l\in\mathbb{N}$ and $C>0$ such that for any $u\in L^2(\mathbb{R}^n)$,*

$$\|p(x,D)u\|\leqslant C\|p\|_{\mathscr{B}^\sigma\Psi^0,l}\|u\|.$$

(ii) *Suppose $\sigma>1$ and $p_m(x,\xi)\in\mathscr{B}^\sigma\Psi^m(\mathbb{R}^n)$, ($m=0,1$). Set*

$$q(x,\xi)=p_0(x,\xi)p_1(x,\xi),\quad r(x,\xi)=\overline{p_1(x,\xi)}.$$

Let $p_1(x,D)^\star$ be the formal adjoint of $p(x,D)$. Then, there exist $l\in\mathbb{N}$ and $C>0$ such that for any $u\in L^2(\mathbb{R}^n)$,

$$\|(p_0(x,D)p_1(x,D)-q(x,D))u\|\leqslant C\|p_0\|_{\mathscr{B}^\sigma\Psi^0,l}\|p_1\|_{\mathscr{B}^\sigma\Psi^1,l}\|u\|,$$

$$\|(p_1(x,D)p_0(x,D)-q(x,D))u\| \leqslant C\|p_0\|_{\mathscr{B}^\sigma\Psi^0,l}\|p_1\|_{\mathscr{B}^\sigma\Psi^1,l}\|u\|,$$
$$\|(p_1(x,D)^\star-r(x,D))u\| \leqslant C\|p_1\|_{\mathscr{B}^\sigma\Psi^1,l}\|u\|.$$

(iii) *Let* $p(x,\xi)=[p_{jk}(x,\xi)]_{j,k=1,\dots,l}$ *be an* $l\times l$ *matrix. Suppose that* $p_{jk}(x,\xi)\in\mathscr{B}^2\Psi^1(\mathbb{R}^n)$*, and there exists* $\lambda\geqslant 0$ *such that*

$$p(x,\xi)+{}^t\overline{p(x,\xi)}\geqslant 0 \quad \textit{for} \quad |\xi|\geqslant\lambda.$$

Then, there exist $l\in\mathbb{N}$ *and* $C>0$ *such that for any* $U\in(L^2(\mathbb{R}^n))^l$*,*

$$\operatorname{Re}(p(x,D)U,U)\geqslant -C\sum_{j,k=1}^{l}\|p_{jk}\|_{\mathscr{B}^2\Psi^1,l}\|U\|^2. \tag{8}$$

For the proof of Lemma 4, see [2] and references therein.

In the proof of Theorem 1, we construct a symbol by using functions of the form

$$\phi_{1,\varepsilon}(t,x_1)=\int_{\mathbb{R}}|\langle D_2\rangle^{1/2+\sigma}u_\varepsilon(t,x_1,x_2)|^2dx_2,$$
$$\phi_{2,\varepsilon}(t,x_2)=\int_{\mathbb{R}}|\langle D_1\rangle^{1/2+\sigma}u_\varepsilon(t,x_1,x_2)|^2dx_1,$$

where σ is a small positive number, $\langle D_j\rangle=(1-\partial_j^2)^{1/2}$, and u_ε is a solution to (5)-(6). We will use the following properties of $\phi_{1,\varepsilon}$ and $\phi_{2,\varepsilon}$.

Lemma 5. *Let* $s>3$*, and let* $u_\varepsilon\in C([0,T_\varepsilon];H^s(\mathbb{R}^2))$ *be a solution to* (5)-(6)*. Pick up* $\sigma\in(0,1)$ *so that* $s\geqslant 3+3\sigma$*. Then,*

$$\phi_{1,\varepsilon}(t,y),\phi_{2,\varepsilon}(t,y)\in C([0,T_\varepsilon];\mathscr{B}^{2+\sigma}(\mathbb{R})), \tag{9}$$

$$\int_{-\infty}^{y}\phi_{1,\varepsilon}(t,z)dz,\int_{-\infty}^{y}\phi_{2,\varepsilon}(t,z)dz\in C^1([0,T_\varepsilon];\mathscr{B}^{\sigma}(\mathbb{R})). \tag{10}$$

Proof. Firstly, we evaluate $(1-\partial_1^2)\phi_{1,\varepsilon}(t,x_1)$. Using the Fourier inversion formula on x_1, the Plancherel-Perseval formula on x_2, and the Schwarz inequality on ξ_1, we deduce

$$\begin{aligned}
&|(1-\partial_1^2)\phi_{1,\varepsilon}(t,x_1)|\\
&\leqslant 2\int_{\mathbb{R}}|\langle D_1\rangle^2\langle D_2\rangle^{1/2+\sigma}u_\varepsilon(t,x_1,x_2)|^2dx_2\\
&=C\int_{\mathbb{R}}\left|\int_{\mathbb{R}}e^{ix_1\xi_1}\langle\xi_1\rangle^2\langle\xi_2\rangle^{1/2+\sigma}\hat{u}_\varepsilon(t,\xi_1,\xi_2)d\xi_1\right|^2d\xi_2\\
&\leqslant C\int_{\mathbb{R}}\left|\int_{\mathbb{R}}\langle\xi_1\rangle^2\langle\xi_2\rangle^{1/2+\sigma}|\hat{u}_\varepsilon(t,\xi_1,\xi_2)|d\xi_1\right|^2d\xi_2\\
&\leqslant C\int_{\mathbb{R}}\left(\int_{\mathbb{R}}\langle\xi_1\rangle^{-1-2\sigma}d\xi_1\right)\\
&\qquad\qquad\times\left(\int_{\mathbb{R}}|\langle\xi_1\rangle^{5/2+\sigma}\langle\xi_2\rangle^{1/2+\sigma}\hat{u}_\varepsilon(t,\xi_1,\xi_2)|^2d\xi_1\right)d\xi_2\\
&\leqslant C\int_{\mathbb{R}}\int_{\mathbb{R}}|\langle\xi_1\rangle^{5/2+\sigma}\langle\xi_2\rangle^{1/2+\sigma}\hat{u}_\varepsilon(t,\xi_1,\xi_2)|^2d\xi_1d\xi_2\\
&\leqslant C\|u_\varepsilon(t)\|_{3+2\sigma}^2.
\end{aligned} \tag{11}$$

Secondly, we evaluate

$$\begin{aligned}
&\partial_1^2\phi_{1,\varepsilon}(t,x_1)-\partial_1^2\phi_{1,\varepsilon}(t,y_1)\\
&=2\operatorname{Re}\int_{\mathbb{R}}\langle\xi_2\rangle^{1/2+\sigma}\partial_1^2u_\varepsilon(t,x_1,x_2)\langle\xi_2\rangle^{1/2+\sigma}\overline{u_\varepsilon(t,x_1,x_2)}dx_2
\end{aligned}$$

$$
\begin{aligned}
&-2\operatorname{Re}\int_{\mathbb{R}}\langle\xi_2\rangle^{1/2+\sigma}\partial_1^2u_\varepsilon(t,y_1,x_2)\langle\xi_2\rangle^{1/2+\sigma}\overline{u_\varepsilon(t,y_1,x_2)}dx_2\\
&+2\int_{\mathbb{R}}\langle\xi_2\rangle^{1/2+\sigma}\partial_1u_\varepsilon(t,x_1,x_2)\langle\xi_2\rangle^{1/2+\sigma}\partial_1\overline{u_\varepsilon(t,x_1,x_2)}dx_2\\
&-2\int_{\mathbb{R}}\langle\xi_2\rangle^{1/2+\sigma}\partial_1u_\varepsilon(t,y_1,x_2)\langle\xi_2\rangle^{1/2+\sigma}\partial_1\overline{u_\varepsilon(t,y_1,x_2)}dx_2.
\end{aligned}
$$

Note that $|e^{ix_1\xi_1}-e^{iy_1\xi_1}|\leqslant 2|\xi_1|^\sigma|x_1-y_1|^\sigma$. In the same way as (11), we deduce

$$
\begin{aligned}
&|\partial_1^2\phi_{1,\varepsilon}(t,x_1)-\partial_1^2\phi_{1,\varepsilon}(t,y_1)|\\
&\leqslant C\max_{k=0,1,2}\int_{\mathbb{R}}\left|\int_{\mathbb{R}}(e^{ix_1\xi_1}-e^{iy_1\xi_1})\xi_1^k\langle\xi_2\rangle^{1/2+\sigma}\hat{u}_\varepsilon(t,\xi)d\xi_1\right|\\
&\qquad\qquad\times\left(\int_{\mathbb{R}}|\langle\xi\rangle^{5/2+\sigma}\hat{u}_\varepsilon(t,\xi)|d\xi_1\right)d\xi_2\\
&\leqslant C|x_1-y_1|^\sigma\int_{\mathbb{R}}\left(\int_{\mathbb{R}}|\langle\xi\rangle^{5/2+2\sigma}\hat{u}_\varepsilon(t,\xi)|d\xi_1\right)^2d\xi_2\\
&\leqslant C|x_1-y_1|^\sigma\int_{\mathbb{R}}\int_{\mathbb{R}}|\langle\xi\rangle^{3+3\sigma}\hat{u}_\varepsilon(t,\xi)|^2d\xi_1d\xi_2\\
&=C\|u_\varepsilon(t)\|_{3+3\sigma}^2|x_1-y_1|^\sigma.
\end{aligned}\tag{12}
$$

Combining (11) and (12), we obtain (9).

Next, we show (10). Set $p_\varepsilon(\partial)=p(\partial)-\varepsilon\Delta$ for short. Note that $p_0(\xi)=p_0'(\xi)\cdot\xi/3$. Using the integration by parts, we deduce

$$
\begin{aligned}
&\partial_t\int_{-\infty}^{x_1}\phi_{1,\varepsilon}(t,y_1)dy_1\\
&=2\operatorname{Re}\int_{-\infty}^{x_1}\int_{\mathbb{R}}\langle D_2\rangle^{1/2+\sigma}\partial_tu_\varepsilon(t,y_1,x_2)\langle D_2\rangle^{1/2+\sigma}\overline{u_\varepsilon(t,y_1,x_2)}dy_1dx_2\\
&=-2\operatorname{Re}\int_{-\infty}^{x_1}\int_{\mathbb{R}}p_\varepsilon(\partial)\langle D_2\rangle^{1/2+\sigma}u_\varepsilon(t,y_1,x_2)\langle D_2\rangle^{1/2+\sigma}\overline{u_\varepsilon(t,y_1,x_2)}dy_1dx_2\\
&+2\operatorname{Re}\int_{-\infty}^{x_1}\int_{\mathbb{R}}\sum_{j=0}^{3}\langle D_2\rangle^{1/2+\sigma}a_jf_j(u_\varepsilon)\langle D_2\rangle^{1/2+\sigma}\overline{u_\varepsilon}dy_1dx_2\\
&=-\frac{2}{3}\operatorname{Re}\int_{-\infty}^{x_1}\int_{\mathbb{R}}p_0'(\partial)\cdot\partial\langle D_2\rangle^{1/2+\sigma}u_\varepsilon(t,y_1,x_2)\langle D_2\rangle^{1/2+\sigma}\overline{u_\varepsilon(t,y_1,x_2)}dy_1dx_2\\
&+2\operatorname{Re}\int_{-\infty}^{x_1}\int_{\mathbb{R}}(p_\varepsilon(\partial)-p_0(\partial))\langle D_2\rangle^{1/2+\sigma}u_\varepsilon\langle D_2\rangle^{1/2+\sigma}\overline{u_\varepsilon}dy_1dx_2\\
&+2\operatorname{Re}\int_{-\infty}^{x_1}\int_{\mathbb{R}}\sum_{j=0}^{3}\langle D_2\rangle^{1/2+\sigma}a_jf_j(u_\varepsilon)\langle D_2\rangle^{1/2+\sigma}\overline{u_\varepsilon}dy_1dx_2\\
&=-\frac{2}{3}\operatorname{Re}\int_{\mathbb{R}}\frac{\partial p_0}{\partial\xi_1}(\partial)\langle D_2\rangle^{1/2+\sigma}u_\varepsilon(t,x_1,x_2)\langle D_2\rangle^{1/2+\sigma}\overline{u_\varepsilon(t,x_1,x_2)}dx_2\\
&+\frac{2}{3}\operatorname{Re}\int_{-\infty}^{x_1}\int_{\mathbb{R}}p_0'(\partial)\langle D_2\rangle^{1/2+\sigma}u_\varepsilon\cdot\partial\langle D_2\rangle^{1/2+\sigma}\overline{u_\varepsilon}dy_1dx_2\\
&+2\operatorname{Re}\int_{-\infty}^{x_1}\int_{\mathbb{R}}(p_\varepsilon(\partial)-p_0(\partial))\langle D_2\rangle^{1/2+\sigma}u_\varepsilon\langle D_2\rangle^{1/2+\sigma}\overline{u_\varepsilon}dy_1dx_2
\end{aligned}
$$

$$+ 2\,\mathrm{Re}\int_{-\infty}^{x_1}\int_{\mathbb{R}}\sum_{j=0}^{3}\langle D_2\rangle^{1/2+\sigma}a_j f_j(u_\varepsilon)\langle D_2\rangle^{1/2+\sigma}\overline{u_\varepsilon}dy_1dx_2.$$

Then, we have

$$\begin{aligned}\left|\partial_t\int_{-\infty}^{x_1}\phi_{1,\varepsilon}(t,y_1)dy_1\right| &\leqslant C\int_{\mathbb{R}}|\langle D\rangle^{5/2+\sigma}u_\varepsilon(t,x_1,x_2)|^2dx_2\\ &\quad + C\|u_\varepsilon(t)\|^2_{5/2+\sigma} + C\sum_{j=0}^{3}\|f_j(u_\varepsilon(t))\|_{1/2+\sigma}\|u_\varepsilon(t)\|_{1/2+\sigma}\\ &\leqslant C(\|u_\varepsilon(t)\|^2_{3+2\sigma}+\|u_\varepsilon(t)\|^4_{3+2\sigma}).\end{aligned}\tag{13}$$

Similarly, we can get

$$\left|\partial_t\int_{y_1}^{x_1}\phi_{1,\varepsilon}(t,z_1)dz_1\right| \leqslant C(\|u_\varepsilon(t)\|^2_{3+3\sigma}+\|u_\varepsilon(t)\|^4_{3+3\sigma})|x_1-y_1|^\sigma.\tag{14}$$

Combining (13) and (14), we obtain (10). □

4. A LINEAR PSEUDODIFFERENTIAL SYSTEM

As is considered in [2], it is very convenient to establish the H^s-well-posedness of the initial value problem for a 2×2 system related to the system for (1) and its complex conjugate. Consider the initial value problem of the form

$$LU = F(t,x) \quad \text{in} \quad (0,T)\times\mathbb{R}^2,\tag{15}$$

$$U(0,x) = U_0(x) \quad \text{in} \quad \mathbb{R}^2,\tag{16}$$

where $U(t,x)$ is a $\mathbb{C}^2$-valued unknown function, $F(t,x)$ and $U_0(x)$ are given $\mathbb{C}^2$-valued functions, $T>0$,

$$L = I(\partial_t + p_0(\partial) - \varepsilon\Delta) + iJp_1(\partial) + A(t) + B(t),$$

$p_0(\xi)$ is same as that of (1), $p_1(\xi)$ is a homogeneous real polynomial of degree two, $\varepsilon \geqslant 0$,

$$I = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \quad J = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix},$$

$$A(t) = \begin{bmatrix}a_{11}(t,x,D) & a_{12}(t,x,D)\\ a_{21}(t,x,D) & a_{22}(t,x,D)\end{bmatrix}, B(t) = \begin{bmatrix}B_{11}(t) & B_{12}(t)\\ B_{21}(t) & B_{22}(t)\end{bmatrix},$$

$a_{jk}(t,x,\xi)\in C([0,T];\mathscr{B}^{2+\sigma}\Psi^1(\mathbb{R}^2))$, ($\sigma>0$), $B_{jk}(t)$ and $[\langle D\rangle^s, B_{jk}(t)]$, ($s\in[0,1]$) are L^2-bounded. Here we state the definition of the well-posedness.

Definition 2. Let $s\in\mathbb{R}$. The initial value problem (15)-(16) is said to be H^s-well-posed if for any $U_0\in(H^s(\mathbb{R}^2))^2$ and $F\in(L^1(0,T;H^s(\mathbb{R}^2)))^2$, (15)-(16) possesses a unique solution $U\in(C([0,T];H^s(\mathbb{R}^2)))^2$.

We give a sufficient condition of H^s-well-posedness used later.

Lemma 6. *Suppose* $s\in[0,1]$ *and* (3). *If there exists* $\phi(t,y)\in C([0,T];\mathscr{B}^{2+\sigma}(\mathbb{R}))$ *such that* $\phi(t,y)\geqslant 0$,

$$\sup_{t\in[0,T]}\int_{\mathbb{R}}\phi(t,y)dy + \sup_{\substack{t\in[0,T]\\ x_1\in\mathbb{R}}}\left|\int_{-\infty}^{x_1}\partial_t\phi(t,y)dy\right| < +\infty,$$

$$\sum_{j=1,2}|\mathrm{Re}\,a_{jj}(t,x_1,x_2,\xi)| + \sum_{j+k=3}|a_{jk}(t,x_1,x_2,\xi)| \leqslant \phi(t,x_\nu)|\xi|$$

for $(t,x)=(t,x_1,x_2)\in[0,T]\times\mathbb{R}^2$, $\nu=1,2$ *and* $|\xi|\geqslant 1$, *then* (15)-(16) *is* H^2*-well-posed.*

Proof. Lemma 6 follows from energy estimates and duality argument. We show the forward L^2-estimate for $LU = F$, and omit the backward energy inequality for $L^\star U = F$. Set $K(t) = Ik(t,x,D)$, $k(t,x,\xi) = e^{\gamma(t,x,\xi)}$, $K'(t) = Ik'(t,x,,D)$, $k'(t,x,\xi) = e^{-\gamma(t,x,\xi)}$,

$$\gamma(t,x,\xi) = \sum_{j=1,2} \int_{-\infty}^{x_j} \phi(t,y)dy \frac{\partial p_0}{\partial \xi_j}(\xi)|p_0'(\xi)|^{-2}|\xi|\chi\left(\frac{\xi}{\lambda}\right),$$

where λ is a positive constant determined later, and $\chi(\xi){\in}C^\infty(\mathbb{R})$ satisfying $\chi(\xi) = 1$ for $|\xi| \geqslant 1$ and $\chi(\xi) = 0$ for $|\xi| \leqslant 1/2$. Since

$$\sigma(K'(t)K(t))(x,\xi),\ \sigma(K(t)K'(t))(x,\xi) = I + O(\lambda^{-3}),$$

there exists $\lambda_0 > 0$ such that $K(t)$ is invertible on $(L^2(\mathbb{R}^2))^2$ for $\lambda \geqslant \lambda_0$. Fix $\lambda \geqslant 1 + \lambda_0$. Note that

$$\sigma([k(t,x,D), p_0(\partial)] - Q_0(t)k(t,x,D))(x,\xi) \in C([0,T]; \mathscr{B}^{1+\sigma}\Psi^0(\mathbb{R}^2)),$$

$$\begin{aligned}
\sigma(Q_0(t))(x,\xi) &= \partial e^{\gamma(t,x,\xi)} \cdot p_0'(\xi) e^{-\gamma(t,x,\xi)} \\
&= \sum_{j=1,2} \phi(t,x_j) \left|\frac{\partial p_0}{\partial \xi_j}(\xi)\right|^2 \frac{|\xi|}{|p_0'(\xi)|^2} \chi\left(\frac{\xi}{\lambda}\right).
\end{aligned}$$

Set $Q(t) = IQ_0(t) + A(t)$ for short. Applying $K(t)$ to L, we deduce

$$K(t)L = \{I(\partial_t + p_0(\partial) - \varepsilon\Delta) + iJp_1(p) + Q(t) + R(t)\}K(t),$$

$$\begin{aligned}
R(t) = \Bigg\{ &\frac{\partial K}{\partial t}(t) + I\Big([k(t,x,D), p_0(\partial)] - Q_0(t)k(t,x,D)\Big) \\
&+ iJ[k(t,x,D), p_1(\partial)] - \varepsilon I[k(t,x,D), \Delta] \\
&+ \Big[[k(t,x,D), a_{jk}(t,x,D)]\Big]_{j,k=1,2} + K(t)B(t)\Bigg\} K(t)^{-1}.
\end{aligned} \tag{17}$$

In view of the hypothesis,

$$\sigma(Q(t))(x,\xi) + \sigma(Q(t)^\star)(x,\xi) \geqslant 0 \quad \text{for} \quad |\xi| \geqslant \lambda,$$

and $R(t)$ is L^2-bounded. Then, we deduce

$$\begin{aligned}
\frac{d}{dt}\|K(t)U(t)\|^2 &= 2\operatorname{Re}(\partial_t K(t)U(t), K(t)U(t)) \\
&= 2\operatorname{Re}((-Ip_0(\partial) - iJp_1(\partial))K(t)U(t), K(t)U(t)) \\
&+ 2\operatorname{Re}((\varepsilon I\Delta - Q(t))K(t)U(t), K(t)U(t)) \\
&- 2\operatorname{Re}(R(t)K(t)U(t), K(t)U(t)) + 2\operatorname{Re}(K(t)F(t), K(t)U(t)) \\
&= -2\varepsilon\|\partial K(t)U(t)\|^2 - 2\operatorname{Re}(Q(t)K(t)U(t), K(t)U(t)) \\
&- 2\operatorname{Re}(R(t)K(t)U(t), K(t)U(t)) + 2\operatorname{Re}(K(t)F(t), K(t)U(t)) \\
&\leqslant -2\operatorname{Re}(Q(t)K(t)U(t), K(t)U(t)) \\
&+ 2C\|K(t)U(t)\|^2 + 2\|K(t)F(t)\|\|K(t)U(t)\|.
\end{aligned}$$

By the sharp Gårding inequality (8),

$$\frac{d}{dt}\|K(t)U(t)\|^2 \leqslant 2C_0\|K(t)U(t)\|^2 + 2\|K(t)F(t)\|\|K(t)U(t)\|,$$

which becomes

$$\frac{d}{dt}\|K(t)U(t)\| \leqslant C_0\|K(t)U(t)\| + \|K(t)F(t)\|. \tag{18}$$

Integrating (18) over $[0,t]$, we obtain

$$\|K(t)U(t)\| \leqslant e^{C_0 t}\|K(0)U_0\| + \int_0^t e^{C_0(t-\tau)}\|K(\tau)F(\tau)\|,$$

which is a desired energy inequality.

Let $s \in (0,1]$. Applying $I\langle D\rangle^s$ to (15), we have

$$(L+\tilde{A}_s(t))\langle D\rangle^s U = \tilde{B}_s(t)U + \langle D\rangle^s F(t,x),$$

$$\tilde{A}_s(t) = \begin{bmatrix} [\langle D\rangle^s, a_{11}(t,x,D)] & [\langle D\rangle^s, a_{12}(t,x,D)] \\ [\langle D\rangle^s, a_{21}(t,x,D)] & [\langle D\rangle^s, a_{22}(t,x,D)] \end{bmatrix} \langle D\rangle^{-s},$$

$$\tilde{B}_s(t) = -\begin{bmatrix} [\langle D\rangle^s, B_{11}(t)] & [\langle D\rangle^s, B_{12}(t)] \\ [\langle D\rangle^s, B_{21}(t)] & [\langle D\rangle^s, B_{22}(t)] \end{bmatrix}.$$

$\tilde{A}_s(t)$ and $\tilde{B}_s(t)$ are L^2-bounded. If (15)-(16) is L^2-well-posed, then The initial value problem for a modified system

$$(L+\tilde{A}_s(t))U = F(t,x) \tag{19}$$

is also L^2-well-posed. The L^2-well-posedness of (15)-(16) and (19)-(16) implies the H^s-well-posedness of (15)-(16). This completes the proof. $\square$

5. Energy method

In this section we prove Theorem 1. Suppose $s > 3$. Let $u_\varepsilon \in C([0,T_\varepsilon]; H^s(\mathbb{R}^2))$ be a unique solution to (5)-(6). Firstly, we construct a solution $u \in L^\infty(0,T;H^s(\mathbb{R}^2))$ with some $T > 0$ by compactness argument. More precisely, we show that there exists $T > 0$ such that $T_\varepsilon \geqslant T$ for any $\varepsilon > 0$, and $\{u_\varepsilon\}_{\varepsilon>0}$ is bounded in $L^\infty(0,T;H^s(\mathbb{R}^2))$.

Let $\chi(\xi)$ be the same as in Section 4. We split R_1 into the principal part and the remainder part by

$$R_1 = r_0(D) + \tilde{R}_1, \quad r_0(\xi)\frac{i\xi}{|\xi|}\chi(\xi), \quad \tilde{R}_1 = R_1(1-\chi(D)).$$

Then,

$$r_0(\xi) \in \Psi^0(\mathbb{R}^2) = \bigcap_{k=1}^{\infty} \mathscr{B}^k\Psi^0(\mathbb{R}^2), \quad \|\tilde{R}_1 v\|_m \leqslant C_m\|v\|$$

for any $m \geqslant 0$ and $v \in L^2(\mathbb{R}^2)$. Since $p(i\xi)$ is pure-imaginary,

$$p(\partial) = p_0(\partial) + ip_1(\partial) + p_2(\partial) + ip_3,$$

where $p_j(\xi)$ is a homogeneous real polynomial of degree $3-j$. Set $\theta = s - [s]$ for short. Let α be a multi-index satisfying $|\alpha| = [s]$. Applying $\langle D\rangle^\theta\partial^\alpha$ to (5), we have

$$\begin{aligned} &(\partial_t + p_0(\partial) + ip_1(\partial) - \varepsilon\Delta)\langle D\rangle^\theta\partial^\alpha u_\varepsilon \\ &\quad + a_{1,\varepsilon}(t,x,D)\langle D\rangle^\theta\partial^\alpha u_\varepsilon + a_{2,\varepsilon}(t,x,D)\overline{\langle D\rangle^\theta\partial^\alpha u_\varepsilon} = f_{\varepsilon,\alpha}, \end{aligned}$$

$$\begin{aligned} a_{1,\varepsilon}(t,x,\xi) &= -i|u_\varepsilon(t,x)|^2(a_0 r_0(\xi) + a_1)\xi_1 + ip_2(\xi), \\ a_{2,\varepsilon}(t,x,\xi) &= -iu_\varepsilon(t,x)^2(a_0 r_0(\xi) + a_2)\xi_1, \end{aligned}$$

$$\begin{aligned} f_{\varepsilon,\alpha} = &\left\{\sum_{j=0}^{3} a_j\langle D\rangle^\theta\partial^\alpha f_j(u_\varepsilon)\right. \\ &\quad \left. - |u_\varepsilon|^2(a_0R_1 + a_1)\partial_1\langle D\rangle^\theta\partial^\alpha u_\varepsilon - u_\varepsilon^2(a_0R_1 + a_1)\partial_1\langle D\rangle^\theta\partial^\alpha \bar{u}_\varepsilon\right\} \\ &+ a_0|u_\varepsilon|^2\tilde{R}_1\partial_1\langle D\rangle^\theta\partial^\alpha u_\varepsilon + a_0 u_\varepsilon^2\tilde{R}_1\partial_1\langle D\rangle^\theta\partial^\alpha\bar{u}_\varepsilon \\ &- [\langle D\rangle^\theta, |u_\varepsilon|^2(a_0R_1 + a_1)\partial_1]\partial^\alpha u_\varepsilon \end{aligned}$$

$$-[\langle D\rangle^\theta, u_\varepsilon^2(a_0R_1+a_1)\partial_1]\partial^\alpha\bar{u}_\varepsilon - ip_3\langle D\rangle^\theta\partial^\alpha u_\varepsilon.$$

Since the highest order of differentiation in $f_{\varepsilon,\alpha}$ is $s=\theta+|\alpha|$,

$$\|f_{\varepsilon,\alpha}(t)\| \leqslant C(\|u_\varepsilon(t)\|_s + \|u_\varepsilon(t)\|_s^3).$$

If we set

$$U_{\varepsilon,\alpha}=\begin{bmatrix}\langle D\rangle^\theta\partial^\alpha u_\varepsilon\\ \overline{\langle D\rangle^\theta\partial^\alpha u_\varepsilon}\end{bmatrix},\quad U_{\alpha,0}=\begin{bmatrix}\langle D\rangle^\theta\partial^\alpha u_0\\ \overline{\langle D\rangle^\theta\partial^\alpha u_\varepsilon}\end{bmatrix},\quad F_{\varepsilon,\alpha}=\begin{bmatrix}f_{\varepsilon,\alpha}\\ \overline{f_{\varepsilon,\alpha}}\end{bmatrix},$$

$$\sigma(A_\varepsilon(t))(x,\xi)=\begin{bmatrix}a_{1,\varepsilon}(t,x,\xi) & a_{2,\varepsilon}(t,x,\xi)\\ \overline{a_{2,\varepsilon}(t,x,-\xi)} & \overline{a_{1,\varepsilon}(t,x,-\xi)}\end{bmatrix},$$

then $U_{\varepsilon,\alpha}$ solves

$$\Big\{I(\partial_t+p_0(\partial)-\varepsilon\Delta)+iJp_1(\partial)+A_\varepsilon(t)\Big\}U_{\varepsilon,\alpha}=F_{\varepsilon,\alpha} \quad \text{in} \quad (0,T_\varepsilon)\times\mathbb{R}^2, \tag{20}$$

$$U_{\varepsilon,\alpha}(0,x)=U_{\alpha,0}(x) \quad \text{in} \quad \mathbb{R}^2.$$

Let $\phi_{1,\varepsilon}$ and $\phi_{2,\varepsilon}$ be the functions introduced in Section 3. Since

$$|\operatorname{Re} a_{1,\varepsilon}(t,x,\xi)|+|a_{2,\varepsilon}(t,x,\xi)| \leqslant (2|a_0|+|a_1|+|a_2|)|u_\varepsilon(t,x)|^2|\xi_1|,$$

there exists $C_0>0$ which is independent of $\varepsilon>0$, such that

$$2|\operatorname{Re} a_{1,\varepsilon}(t,x,\xi)|+2|a_{2,\varepsilon}(t,x,\xi)| \leqslant C_0|\xi|\min\{\phi_{1,\varepsilon}(t,x_1),\phi_{2,\varepsilon}(t,x_2)\}. \tag{21}$$

Set

$$\phi_\varepsilon(t,y)=C_0\phi_{1,\varepsilon}(t,y)+C_0\phi_{2,\varepsilon}(t,y),$$

$$\gamma_\varepsilon(t,x,\xi)=\sum_{j=1,2}\int_{-\infty}^{x_j}\phi_\varepsilon(t,y)dy\frac{\partial p_0}{\partial\xi_j}(\xi)\frac{|\xi|}{|p_0'(\xi)|^2}\chi(\xi),$$

$$\sigma(K_\varepsilon(t))(x,\xi)=Ie^{\gamma_\varepsilon(t,x,\xi)},\quad \sigma(K_\varepsilon'(t))(x,\xi)=Ie^{-\gamma_\varepsilon(t,x,\xi)}.$$

In view of (9), (10) and (21), ϕ_ε satisfies the conditions in Lemma 6 for (20).

We evaluate

$$N_\varepsilon(t)=\sum_{|\alpha|=[s]}\|K_\varepsilon(t)U_{\varepsilon,\alpha}(t)\|+\|u_\varepsilon(t)\|_{s-1}.$$

Since $N_\varepsilon(0)$ is independent of $\varepsilon>0$, set $M=N_\varepsilon(0)$ for short. Here we introduce

$$T_\varepsilon^\star=\sup\{T>0|N_\varepsilon(t)\leqslant 2M \text{ for } t\in[0,T]\}.$$

Since

$$\sigma(K_\varepsilon'(t)K_\varepsilon(t))(x,\xi),\sigma(K_\varepsilon(t)K_\varepsilon'(t))(x,\xi)=I+O(\langle\xi\rangle^{-3}),$$

there exists $C_M>1$ which is independent of $\varepsilon>0$, such that

$$C_M^{-1}\|u_\varepsilon(t)\|_s \leqslant N_\varepsilon(t)\leqslant C_M\|u_\varepsilon(t)\|_s \quad \text{for} \quad t\in[0.T_\varepsilon^\star].$$

Applying $K_\varepsilon(t)$ to (20), we have

$$\Big\{I(\partial_t+p_0(\partial)-\varepsilon\Delta)+iJ\partial_1(\partial)+Q_\varepsilon(t)\Big\}K_\varepsilon(t)U_{\varepsilon,\alpha}+R_\varepsilon(t)U_{\varepsilon,\alpha}=K_\varepsilon(t)F_{\varepsilon,\alpha},$$

$$Q_\varepsilon(t)=IQ_{\varepsilon,0}(t)+A_\varepsilon(t),$$

$$\sigma(Q_{\varepsilon,0}(t))(x,\xi)=\sum_{j=1,2}\phi_\varepsilon(t,x_j)\left|\frac{\partial p_0}{\partial\xi_j}(\xi)\right|^2\frac{|\xi|}{|p_0'(\xi)|^2}\chi(\xi),$$

$R_\varepsilon(t)$ corresponds to $R(t)K(t)$ in (17). In view of Lemmas 4 and 5, we get

$$\|R_\varepsilon(t)U_{\varepsilon,\alpha}(t)\| \leqslant CM^2N_\varepsilon(t) \quad \text{for} \quad t\in[0,T_\varepsilon^\star].$$

In the same way as the energy estimate in Section 4, we deduce

$$\frac{d}{dt}\|K_\varepsilon(t)U_{\varepsilon,\alpha}(t)\|^2 \leqslant -2\operatorname{Re}(Q_\varepsilon(t)K_\varepsilon(t)U_{\varepsilon,\alpha}(t),K_\varepsilon(t)U_{\varepsilon,\alpha}(t))$$

$$
\begin{aligned}
&+ 2\Big(\|R_\varepsilon(t)U_{\varepsilon,\alpha}(t)\| + \|K_\varepsilon(t)F_{\varepsilon,\alpha}(t)\|\Big)\|K_\varepsilon(t)U_{\varepsilon,\alpha}(t)\| \\
&\leqslant 2C_1M^2N_\varepsilon(t)^2
\end{aligned}
$$

for $t \in [0, T_\varepsilon^\star]$, where $C_1 > 0$ depends only on M. Then, we have

$$
\|K_\varepsilon(t)U_{\varepsilon,\alpha}(t)\| \leqslant \|K_\varepsilon(0)U_{\alpha,0}\| + C_1M^2\int_0^t N_\varepsilon(\tau)d\tau. \tag{22}
$$

Using (7), we get

$$
\begin{aligned}
\|u_\varepsilon(t)\|_{s-1} &\leqslant \|u_0\|_{s-1} + \sum_{j=0}^{3}|a_j|\int_0^t \|f_j(u(\tau))\|_{s-1}d\tau \\
&\leqslant \|u_0\|_{s-1} + C_2M^2\int_0^t N_\varepsilon(\tau)d\tau.
\end{aligned} \tag{23}
$$

Combining (22) and (23), we obtain

$$
N_\varepsilon(t) \leqslant N_\varepsilon(0) + C_3M^2\int_0^t N_\varepsilon(\tau)d\tau
$$

for $t \in [0, T_\varepsilon^\star]$, where $C_3 > 0$ is independent of $\varepsilon > 0$. The Gronwall inequality implies that

$$
N_\varepsilon(t) \leqslant M\exp\Big(C_3M^2t\Big) \quad \text{for} \quad t \in [0, T_\varepsilon^\star].
$$

If we set $t = T_\varepsilon^\star$, then $2M \leqslant \exp(C_3M^2T_\varepsilon^\star)$, which gives $T_\varepsilon^\star \geqslant \log 2/C_3M^2$. Set $T = \log 2/C_3M^2$ for short. $\{u_\varepsilon\}_{\varepsilon>0}$ is bounded in $L^\infty(0, T; H^s(\mathbb{R}^2))$. The standard compactness argument shows that there exist a subsequence $\{u_\varepsilon\}$ and u such that

$$
\begin{aligned}
&u_\varepsilon \longrightarrow u \quad \text{in} \quad L^\infty(0, T; H^s(\mathbb{R}^2)) \quad \text{weakly}^\star, \\
&u_\varepsilon \longrightarrow u \quad \text{in} \quad C([0, T]; H^{s-\delta}_{\text{loc}}(\mathbb{R}^2)), \quad (\delta > 0),
\end{aligned}
$$

as $\varepsilon \downarrow 0$. It is easy to see

$$
u \in L^\infty(0, T; H^s(\mathbb{R}^2)) \cap C([0, T]; H^{s-\delta}(\mathbb{R}^2)), \quad (\delta > 0), \tag{24}
$$

and u solves (1)-(2) in the sense of distribution.

Secondly, we prove the uniqueness of solution. Let $u, v \in L^\infty(0, T; H^s(\mathbb{R}^2))$ be solutions to (1) with $u(0) = v(0)$. Set $w = u - v$ for short. Then, $w(0) = 0$, and w solves

$$
(\partial_t + p(\partial) + a_1(t, x, D) + B_1(t))w + (a_2(t, x, D) + B_2(t))\bar{w} = 0, \tag{25}
$$

$$
\begin{aligned}
a_1(t, x, \xi) &= -ia_0|v(t, x)|^2 r_0(\xi)\xi_1 - ia_1|v(t, x)|^2\xi_1, \\
a_2(t, x, \xi) &= -ia_0u(t, x)v(t, x)r_0(\xi)\xi_1 - ia_2v(t, x)^2\xi_1,
\end{aligned}
$$

$$
\begin{aligned}
B_1(t) = &-a_0\Big\{(R_1\partial_1|u|^2) + v[r_0(D), \bar{v}\partial_1] + vR_1(\partial_1\bar{v}) + v\tilde{R}_1\bar{v}\Big\} \\
&- a_1\bar{u}\partial_1u - a_2(u+v)\partial_1\bar{u} - a_3(u+v)\bar{u}, \\
B_2(t) = &-a_0v\Big([r_0(D), u\partial_1] + R_1(\partial_1u) + \tilde{R}_1u\Big) - a_1v\partial_1u - a_3v^3.
\end{aligned}
$$

By Lemma 6, the initial value problem for the system of (25) and its complex conjugate is L^2-well-posed. Thus, ${}^t[w(t), \overline{w(t)}] = 0$.

Lastly, we recover the continuity in the time variable. Let $u \in L^\infty(0, T; H^s(\mathbb{R}^2))$ be a unique solution to (1)-(2). Recall (24). Let α be a multi-index satisfying $|\alpha| = [s]-1$. Set $\theta = s-[s]$ and $u_\alpha = \langle D\rangle^\theta\partial^\alpha u$ for short. It suffices to show $u_\alpha \in C([0, T]; H^1(\mathbb{R}^2))$. Applying $\langle D\rangle^\theta\partial^\alpha$ to (1), we have

$$
(\partial_t + p_0(\partial) + ip_1(\partial) + a_1(t, x, D))u_\alpha + a_2(t, x, D)\overline{u_\alpha} = f_\alpha, \tag{26}
$$

$$a_1(t,x,\xi) = -i|u(t,x)|^2(a_0r_0(\xi)+a_1)\xi_1 + ip_2(\xi),$$
$$a_2(t,x,\xi) = -iu(t,x)^2(a_0r_0(\xi)+a_2)\xi_1,$$

$$\begin{aligned} f_\alpha = \Bigg\{ & \sum_{j=0}^{3} a_j\langle D\rangle^\theta \partial^\alpha f_j(u) \\ & - |u|^2(a_0R_1+a_1)\partial_1\langle D\rangle^\theta\partial^\alpha u - u^2(a_0R_1+a_1)\partial_1\langle D\rangle^\theta\partial^\alpha \bar{u} \Bigg\} \\ & + a_0|u|^2\tilde{R}_1\partial_1\langle D\rangle^\theta\partial^\alpha u + a_0u^2\tilde{R}_1\partial_1\langle D\rangle^\theta\partial^\alpha\bar{u} \\ & - [\langle D\rangle^\theta, |u|^2(a_0R_1+a_1)\partial_1]\partial^\alpha u \\ & - [\langle D\rangle^\theta, u^2(a_0R_1+a_1)\partial_1]\partial^\alpha \bar{u} - ip_3\langle D\rangle^\theta\partial^\alpha u. \end{aligned}$$

It is easy to see that $u_\alpha(0)\in H^1(\mathbb{R}^2)$ and $f_\alpha\in L^1(0,T;H^1(\mathbb{R}^2))$ in the same way as $f_{\varepsilon,\alpha}$. It follows that $u_\alpha\in C([0,T];H^1(\mathbb{R}^2))$ since the initial value problem for the system of (26) and its complex conjugate is H^1-well-posed.

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